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A MODIFIED ADOMIAN'S DECOMPOSITION METHOD†

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A modification of Adomian's method [1–4], based on the use of Padé approximants [5], is proposed. Examples are considered: a non-linear differential equation, a rectangular plate under transverse pressure, and a combination of transverse pressure and longitudinal compression. © 1998 Elsevier Science Ltd. All rights reserved.

The version of the method proposed here turns out to be more effective from the standpoint of practical convergence, and moreover it implies certain theoretical conclusions about the convergence domain of the solution and the rate of convergence. Decomposition, unlike perturbation methods [6–8], does not presume the presence of a small parameter [1–4, 9]—a circumstance that eliminates various limitations such as "weak" non-linearity or "small" deviations of the domain from canonical form. At the same time, in some cases the solutions obtained by the method require generalized summation; in problems of non-linear mechanics this has been tackled successfully using the Padé method of rational-fractional transformation [10–13].

1. We will begin with a simple example. Consider the following ordinary non-linear differential equation with initial condition

$$y'+y^3=0,\ \ y(0)=\tfrac{1}{2} \tag{1.1}$$

The exact solution of this problem is

$$y=(2x+4)^{-1/2} \tag{1.2}$$

The solution of problem (1.1) produced by the decomposition method [1] is

$$y=\frac{1}{2}-\frac{1}{8}x+\frac{3}{64}x^2-\dots \tag{1.3}$$

while that given by the perturbation method [6] is

$$y=\frac{1}{2}-\frac{1}{8}x+\frac{3}{64}x^2-\frac{5}{256}x^3+\dots \tag{1.4}$$

A comparison of the terms of the expansion of the approximations by the methods of decomposition (1.3) and perturbations (1.4) shows that they are identical with those of the Maclaurin expansion of the exact solution (1.2) up to the term of the same degree as the order of the approximation. Obviously, the radii of convergence will also be identical with that of the expansion of the exact solution (which is 2). In this case, then, it would be legitimate to apply meromorphic continuation of the solution according to the Padé scheme [5]. To enlarge the convergence domain, we will use Padé approximants (PAs) of order [0/1] and [1/1]. These are the same for the exact solution and both approximations

$$F[0/1]=\frac{2}{4+x},\ \ F[1/1]=\frac{8+x}{16+6x}$$

Application of these PAs enables one to enlarge the domain within which the results are applicable in practice by a factor of more than 2 with a relative error of 10%.

Analysis of the exact solution and the approximations produced in this example by perturbation and decomposition methods, as well as their PAs, leads one to the following conclusions, which will be rigorously proved in the course of this paper.

1. In both the decomposition method and the perturbation method, the components at powers of the variable, summed over all approximations, are identical with the expansion of the exact solution of the equation in a Maclaurin series.

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2. The convergence domain of the approximations obtained by these methods is a disk about zero with radius equal to the distance to the nearest singular point of the exact solution.
3. For meromorphic continuation of the solution to the nearest essentially singular point of the true solution, it is legitimate to use PAs; this approach considerably improves the convergence.

2. We will now consider the problem in a more general formulation. Consider the following non-linear problem for a system of ordinary differential equations in unknown functions $\{u_i\}_{i=1}^n$ in a domain Ω

$$L_i u_i + R_i(u_1,\ldots,u_n) + N_i(u_i,\ldots,u_n) = g_i,\ i=1,\ldots,n \tag{2.1}$$

$$L_i = \frac{\partial^{k_i}}{\partial\xi^{k_i}},\quad g_i = \sum_{j=0}^{\infty} g_{ij}\xi^j$$

with boundary conditions on the boundary $\partial\Omega$

$$G_j(u_1,\ldots,u_n) = \mu_j,\ j=1,\ldots,k,\ k=k_1+\ldots+k_n, \mu_j = \text{const} \tag{2.2}$$

where L_i is the operator of the highest-order derivative of the ith equation with respect to the independent variable ξ, $\mathbf{R}_i$ is a linear differential operator containing derivatives of lower order, N_i and G_j are non-linear differential operators containing derivatives of lower order and $g_i = g_i(\xi)$ is a continuous function of ξ. We shall also assume that R_i, N_i and G_j vanish when the unknown functions and their derivatives vanish and that $\xi = 0$ is an interior point of Ω. Suppose that we have found a solution $\{\bar{u}_i\}_{i=1}^n$ of problem (2.1), (2.2) whose components are analytic in some neighbourhood of $\xi = 0$ and may therefore be expanded in Taylor series

$$\bar{u}_i = \sum_{k=0}^{\infty} \bar{u}_{ij}\xi^j,\ i=1,\ldots,n \tag{2.3}$$

Considering the u_is and their derivatives as independent arguments, we express R_i, N_i and G_j as generalized multidimensional Taylor series

$$R_i(u_1,\ldots,u_n)+N_i(u_1,\ldots,u_n)=$$

$$=\sum_{j=1}^{n}\sum_{l=0}^{k_j-1} N_{ijl}\frac{\partial^l u_j}{\partial\xi^l}+\frac{1}{2!}\sum_{j,p=1}^{n}\sum_{l=0}^{k_j-1}\sum_{m=0}^{k_p-1} N_{ijplm}\frac{\partial^l u_j}{\partial\xi^l}\frac{\partial^m u_p}{\partial\xi^m}+\ldots$$

$$G_j(u_1,\ldots,u_n)=\sum_{q=1}^{n}\sum_{l=0}^{k_q-1} G_{jql}\frac{\partial^l u_q}{\partial\xi^l}+\frac{1}{2!}\sum_{q,p=1}^{n}\sum_{l=0}^{k_q-1}\sum_{m=0}^{k_p-1} N_{jqplm}\frac{\partial^l u_q}{\partial\xi^l}\frac{\partial^m u_p}{\partial\xi^m}+\ldots \tag{2.4}$$

Substituting expansions (2.3) and (2.4) into the original equation (2.1), we obtain the condition needed to determine $\bar{u}_{ij}$, as an infinite system of equations

$$\bar{u}_{ik_i} = \frac{1}{k_i!}\left(g_{i0} - \sum_{j=1}^{n}\sum_{l=0}^{k_j-1} N_{ijl}\bar{u}_{jl}l! - \frac{1}{2!}\sum_{j,p=1}^{n}\sum_{m=0}^{k_p-1}\sum_{l=0}^{k_j-1} N_{ijplm}\bar{u}_{jl}l!\bar{u}_{pm}m! - \ldots\right) \tag{2.5}$$

$$\bar{u}_{is} = -\frac{(s-k_i)!}{s!}\Bigg(-g_{is} + \sum_{j=1}^{n}\sum_{l=0}^{k_j-1} N_{ijl}\bar{u}_{j(l+s-k_i)}\frac{(l+s-k_i)!}{(s-k_i)!} +$$

$$+\frac{1}{2!}\sum_{j,p=1}^{n}\sum_{l=0}^{k_j-1}\sum_{m=0}^{k_p-1} N_{ijplm}\sum_{q=l}^{l+s-k_i}\bar{u}_{jq}\bar{u}_{p(l+m-q+s-k_i)}\times$$

$$\times\frac{q!}{(q-l)!}\frac{(l+m-q+s-k_i)!}{(l-q+s-k_i)!}+\ldots\Bigg),\quad s=k_i+1,\ k_i+2,\ldots;\ i=1,\ldots,n$$

Thus, each of the components $\bar{u}_{ij}$, beginning with $s = k_i$, is recursively determined, via formulae (2.5), from the components $\bar{u}_{i0}, \ldots, \bar{u}_{i(k_1-1)}$. To determine the latter, we have k boundary conditions, which cannot be separated according to the components of ξ^k.

Substituting expansions (2.3) and (2.4) into boundary conditions (2.2), we obtain

$$\sum_{q=1}^{n}\sum_{l=0}^{k_q-1} G_{igl}\sum_{p=l}^{\infty}\bar{u}_{qp}(\xi^0)\frac{p!}{(p-l)!}+\frac{1}{2!}\sum_{q,p=1}^{n}\sum_{l=0}^{k_q-1}\sum_{m=0}^{k_p-1}G_{iqplm}\sum_{s=l}^{\infty}\bar{u}_{qs}(\xi^0)^{s-l}\frac{s!}{(s-l)!}\times$$

$$\times\sum_{r=m}^{\infty}\bar{u}_{pr}(\xi^0)^{r-m}\frac{r!}{(r-m)!}+...=\mu_j,\ j=1,...,k \tag{2.6}$$

The form of system (2.5) and boundary conditions (2.6) is such that one can also consider an infinite number of unknown functions ($n = \infty$), in which case the number of equations is also infinite ($k = \infty$), but their form remains the same.

If conditions (2.5) and (2.6) are to be sufficient to determine the solution of the boundary-value problem, we must assume that $\{\bar{u}_i\}_{i=1}^{n}$ is unique in some circular neighbourhood of $\xi = 0$ with non-zero radius, which is a natural condition in most mechanical problems. By the uniqueness of the solution and the fact that Eqs (2.5) uniquely define the relationship between the first k_i terms in the expansion of the unknown functions and all the following ones in the construction, the first components of (2.6) are also uniquely defined. Thus, Eqs (2.5) and (2.6) completely and uniquely define the solution of boundary-value problem (2.1), (2.2). We may thus assume that if a solution of system (2.1) with boundary conditions (2.2) exists and is unique and analytic in the neighbourhood of $\xi = 0$, then series (2.3) will be the Maclaurin expansion of the solution, as a function of ξ, if and only if its components satisfy Eqs (2.5) and (2.6).

3. We will now reformulate the initial boundary-value problem as a problem of perturbation theory, introducing an artificial small parameter ε_1 as done by Dorodnitsyn [6] (an analogous scheme in computational mathematics is known as the method of continuation with respect to a parameter).

Consider the solution of the boundary-value problem (2.1), (2.2) by the scheme of perturbation theory with an artificial small parameter ε_1, in the following form

$$L_i u_i = -\varepsilon_1(R_i(u_1,...,u_n)+N_i(u_1,...,u_n)-g_i),\ i=1,...,n \tag{3.1}$$

Note that when $\varepsilon_1 = 1$ the results of applying this perturbation method are the same as those obtained by Adomian's decomposition method [1]. Expressing the required solution as

$$u_i = \sum_{j=0}^{\infty} u_{ij}\varepsilon_1^j,\ i=0,1,... \tag{3.2}$$

substituting expansion (3.2) into Eq. (3.1) and boundary conditions (2.2) and splitting the system in powers of ε_1, we obtain a sequence of limiting boundary-value problems

$$\varepsilon_1^0:\ L_i u_{i0}=0, i=1,...,n;\ G_j(u_{10},...,u_{n0})=\mu_j,\ j=1,...,k \tag{3.3}$$

$$\varepsilon_1^j:\ L_i u_{ij} = -(R_i(u_{1(j-1)},...,u_{n(j-1)})+$$

$$+N_i(u_{10},...,u_{n0},...,u_{1(j-1)},...,u_{n(j-1)})-g_i),\ i=1,...,n \tag{3.4}$$

$$G_j(u_{10},...,u_{n0},...,u_{1(j-1)},...,u_{n(j-1)})=0,\ j=1,...,k$$

where N_i and G_j are the components of Eq. (2.1) and boundary conditions (1.2) for ε_1^j. When expansion (2.4) is substituted into Eqs (2.1) and (2.2), these components may be expressed as generalized Taylor series in the unknown functions and their derivatives.

The components of the solution in the zeroth approximation have the form of polynomials and so

$$u_i = \sum_{j=0}^{\infty}\varepsilon_1^j\sum_{m=0}^{R_j}u_{ijm}\xi^m, i,j=0,1,... \tag{3.5}$$

The boundary-value problems (3.3) and (3.4) yield the coefficients u_{ijm}.

Substituting them into series (3.5) and inverting the order of summation when $\varepsilon_1 = 1$ (this is admissible if the solution is convergent in some neighbourhood of $\xi = 0$ of non-zero radius, which is a natural condition in mechanical problems), we get

$$u_i = \sum_{m=0}^{\infty}\xi^m\sum_{j=0}^{\infty}u_{ijm} = \sum_{m=0}^{\infty}\xi^m u_{im} \tag{3.6}$$

$$u_i = \sum_{m=0}^{\infty} \xi^m \sum_{j=0}^{\infty} u_{ijm} = \sum_{m=0}^{\infty} \xi^m u_{im}$$

$$u_{is} = -\frac{(s-k_i)!}{s!}\left(\sum_{r=1}^{\infty} \sum_{l=0}^{k_r-1} \frac{(s-k_i+2l)!}{(s-k_i)!} N_{irl} u_{r(s-k_i+l)} + \right.$$

$$+\frac{1}{2!} \sum_{q,p=1}^{\infty} \sum_{l=0}^{k_q-1} \sum_{m=0}^{k_p-1} N_{iqplm} \sum_{t=m}^{s+m-k_i} \left\{ \sum_{j=1}^{\infty} \sum_{r=0}^{j-1} u_{p(j-1-r)t} \times \right.$$

$$\left. \left. \times u_{qr(s-k_i+l-t+m)} \right\} \frac{t!}{(t-m)!} \frac{(s-k_i+l-t+m)!}{(s-k_i-t+m)!} + \ldots \right) + \frac{(s-k_i)!}{s!} g_{is} \tag{3.7}$$

$$i = 1, 2, \ldots; \; s = k_i, \; k_i + 1, \ldots$$

After a transformation of the sum in braces in (3.7), analogous to the convolution mapping in the Laplace integral transform [14], expression (3.7) is identical with (2.5), i.e. the solution obtained by the proposed perturbation method satisfies the necessary condition for the components of the Taylor series of the exact solution. Thus, one-dimensional non-linear problems may be effectively solved both by the decomposition method in Adomian's version and by a perturbation method with an analogous scheme; the solution obtained will have the form of the Maclaurin expansion of the solution of the initial problem.

4. The above conclusion contains important information on the domain and the nature of the convergence of the solution. Since the series $\Sigma_{j=0}^{\infty} u_{ijm}$ in (3.7) has the meaning of a coefficient of the Maclaurin series of u_i in powers of ξ, it follows that $\varepsilon_1 = 1$ belongs to the disk of convergence for all ξ that lie inside the disk of convergence with respect to that variable. Therefore, if the solution is convergent as a series in powers of ξ in the neighbourhood of $\xi = 0$, with a non-zero radius of convergence, the properties of power series [14] imply that the sum of the series (3.6) is a continuous function $a(\varepsilon_1)$ which when $\varepsilon_1 = 1$ is identical with the solution of the initial boundary-value problem, for which (3.6) is the Maclaurin expansion. Generalized summation using PAs accomplishes meromorphic continuation of the solution [5], which is a valid in a transformation both with respect to ξ and with respect to ε_1. By virtue of this property, the sequence of PAs of the solution found using the method proposed here with respect to ξ or ε_1, converges uniformly, as the order of the numerator increases, to the true solution in a disk about zero, within which the solution is meromorphic in the appropriate variable, while the total order of the poles is equal to the order of the denominator of the approximants, with the exception of the poles themselves.

Thus, in the limit, the method proposed here yields a solution of the boundary-value in its domain of meromorphicity.

5. Let us consider a scheme for solving two-dimensional problems. Since many problems in the theory of plates and shells are periodic with respect to one of the variables (which we denote by η) or may be reduced to periodic problems, while the expansion of the unknown and given quantities in trigonometric series makes it possible to comply completely with the periodicity conditions and determines the form of the functions and their derivatives as functions of η, we shall treat η as a parameter, thus essentially making the problem one-dimensional.

After expanding the unknown quantities in series of trigonometric functions, substituting them into the initial equations and transforming, we obtain an infinite system of type (2.1) with boundary conditions of type (2.2) with respect to the components of the expansion ($n = k = \infty$). The solution of this boundary-value problem by the method considered in this paper has the form of (3.6), (3.7).

6. We will investigate the practical convergence of the method using model problems. As the first example we consider a linear formulation of the problem of the bending of a rectangular plate, supported on hinges, under uniform pressure. When the approximate solution was constructed, the form of the boundary was artificially perturbed by increasing the length of the plate from τ_0 to τ. The solution obtained by the method proposed here was compared with the exact solution [15] and the error was computed as a percentage of the value of the deflection for the exact solution. When the numerical experiment was carried out, the ratio of the length of the sides of the plate and the amplitude of the perturbation of the form of the boundary were varied. An idea of the dependence of the computation error of the maximum deflection on the parameters, in percentages, may be obtained from Table 1. For comparison, we used PAs of orders [0/1] and [1/1]. Analysis of the numerical data indicates that the approximate solution gives satisfactory accuracy for a broad range of plates. The [0/1] PA is preferable, since it yields the same accuracy or better as the [1/1] PA, with less computation.

Analogous results were obtained when analysing a non-linear model of the deformation of a rectangular plate under transverse pressure, taking membrane forces into account. In this example, besides the parameters introduced previously, the force of uniform longitudinal compression of the plate, N_x, was also varied. The results are shown in Table 2, where the dependence of the relative error on the plate parameters and the value of the longitudinal load, expressed as a percentage, is shown. Note that increasing the value of N_x from a third to two-thirds of the

Table 1

$\frac{\tau_0\pi}{a}$	$\tau/\tau_0 = 1.00$		1.01		1.10	
	[0/1]	[1/1]	[0/1]	[1/1]	[0/1]	[1/1]
1	0.0	0.0	0.2	0.2	9.5	2.5
2	0.1	0.0	0.7	0.6	3.1	7.0
3	0.4	0.7	0.5	0.8	6.9	9.3
4	1.0	0.3	0.5	0.6	6.8	9.7
5	2.1	0.9	1.5	0.3	5.0	8.5
6	3.4	2.3	2.8	1.6	2.8	4.9

Table 2

N_x	$\frac{\tau_0\pi}{a}$	$\tau/\tau_0 = 1.00$		1.01		1.10	
		[0/1]	[1/1]	[0/1]	[1/1]	[0/1]	[1/1]
$\frac{T_*}{3}$	2	0.1	0.0	0.3	0.4	6.8	3.8
	3	0.2	0.1	0.2	0.7	0.8	5.1
	4	0.5	0.9	1.0	0.7	1.1	5.0
	5	1.6	0.2	2.0	0.4	2.1	3.5
$\frac{2T_*}{3}$	2	0.0	0.0	3.0	1.2	60.0	5.0
	3	1.0	0.0	1.3	1.0	25.0	7.0
	4	1.5	0.0	1.5	0.9	23.5	3.1
	5	2.5	1.0	5.0	0.7	29.0	60.0

critical axial compression load T_* did not produce appreciably inferior results. However, to realize the perturbation of the form of the boundary in this case it was necessary to use at least three approximations and PAs of order [1/1].

A characteristic feature of both examples is the increased accuracy of the computations for nearly square plates. This enables one to treat the method as a supplement to traditional asymptotic methods, whose application requires the geometrical dimensions of the construction to be considerably different.

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REFERENCES

1. ADOMIAN, G., A review of the decomposition method in applied mathematics. *J. Math. Anal. Appl.*, 1988, **135**(2), 501–544.
2. ADOMIAN, G., Decomposition solution for Duffing and Van der Pol oscillators. *Int. J. Math. Sci.*, 1986, **9**(4), 731–732.
3. ADOMIAN, G., *Solving Frontier Problems of Physics: The Decomposition Method*. Kluwer, Dordrecht, 1994.
4. BELLOMO, N. and MONACO, R., A comparison between Adomian's decomposition methods and perturbation techniques for nonlinear random differential equations. *J. Math. Anal. Appl.*, 1985, **110**(2), 495–502.
5. BAKER, G. A. and GRAVES-MORRIS, P. R., *Padé Approximants*. Addison-Wesley, Reading MA, 1981.
6. KAYUK, Ya. F., *Some Methods of Expansion with Respect to a Parameter*. Naukova Dumka, Kiev, 1980.
7. NAYFEH, A. H., *Perturbation Methods*. John Wiley, New York, 1973.
8. OBRAZTSOV, I. F., NERUBAILO, B. V. and ANDRIANOV, I. V., *Asymptotic Methods in the Engineering Mechanics of Thin-walled Structures*. Mashinostroyeniye, Moscow, 1991.
9. PSHENICHNOV, G. I., The method of decomposition of the solution of equations and boundary-value problems. *Dokl. Akad. Nauk SSSR*, 1985, **282**(4), 792–794.
10. ANDRIANOV, I. V. and BULANOVA, N. S., A method of improving the convergence of series in perturbation theory in mechanics. *Prikl. Mekh.*, 1984, **20**(5), 119–122.
11. ANDRIANOV, I. V. and IVANKOV, A. O., Application of Padé approximants in the method of introducing a parameter in the investigation of a biharmonic equation with complicated boundary conditions. *Zh. Vychisl. Mat. Mat. Fiz.*, 1987, **27**(2), 296–301.
12. ANDRIANOV, I. V. and IVANKOVA, A. O., The solution of mixed problems of the theory of bending of plates by a modified method of perturbing the form of the boundary conditions. *Dokl. Akad. Nauk UkrSSR, Ser. A*, 1988, 1, 33–36.

13. ANDRIANOV, I. V. and OLEVSKII, V. I., Non-linear deformation and stability of rectangular plates and cylindrical shells with periodic perturbation of the shape of the end. In *Proceedings of the 4th All-Union Conference on "Current Problems of Engineering Mechanics and the Strength of Aircraft"*. Khar'kov Aviation Institute, Khar'kov, 1991, 86.
14. KORN, G. A. and KORN, T. M., *Mathematical Handbook for Scientists and Engineers*. McGraw-Hill, New York, 1968.
15. TIMOSHENKO, S. P. and WOINOWSKI-KRIEGER, S., *Theory of Plates and Shells*. McGraw-Hill, New York, 1959.

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